Representations of the Infinite Matrix Algebra

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Infinite dimensional Lie algebras frequently occur in the modern formulation of the theory of certain nonlinear partial differential equations from mathematical physics. As examples we mention the well-known Korteweg-de Vries equation and its higher dimensional generalization, the Kadomtzev-Petviashvili equation. The relevant Lie algebra for these examples is the collection of $\infty \times \infty$ matrices with complex entries. In this paper we present a survey of the theory of projective representations of this infinite matrix algebra. This greatly helps in the description of a large class of explicit solutions of these partial differential equations (pdes), known as soliton solutions. The ideas and methods used in this area of mathematics are a mixture of a wide variety of physical concepts. Among them are Dirac's classical theory of electrons and positrons but also more modern ideas from two dimensional conformal quantum field theory.

1. Introduction

In 1895 Korteweg and de Vries introduced the following partial differential equation:

$$\frac{\partial f}{\partial t} - \frac{3}{2} f \frac{\partial f}{\partial x} - \frac{1}{4} \frac{\partial^3 f}{\partial x^3} = 0. \tag{1.1}$$

This equation, which is now known under the name KdV-equation, is one of the most celebrated examples of a so-called soliton equation. It was proposed in order to describe the propagation of shallow water waves in a narrow channel. In this context the dependent variable $f = f(x,t)$ describes the height of waves in the channel as a function of a space coordinate x along the channel and a time coordinate t.

It is not too difficult to verify that (1.1) has the solution

$$f(x,t) = 8u^2/(e^{u(x+u^2t+\alpha)} + e^{-u(x+u^2t+\alpha)})^2 = 2u^2 \operatorname{sech}^2 u(x+u^2t+\alpha), \tag{1.2}$$

where u and α are arbitrary (real) parameters. This solution describes the motion of a wave, which moves in time without loss of shape; for $t=0$ it looks like $2u^2\operatorname{sech}^2 u(x+\alpha)$, which is a single 'pulse' peaked at $x=0$, and this profile propagates with a velocity u^2 in the negative x-direction. It is worthwhile to notice that the velocity is proportional to the amplitude of the wave. Such a wave is called a solitary wave or soliton. It had been observed in nature by Russell as early as 1834, and in fact the explanation of Russell's discovery was the motivation for Korteweg and de Vries in introducing (1.1). We refer the interested reader to Newell's book [1] for a beautiful historical overview of the discovery of the soliton.

In the 20th century the KdV-equation was forgotten for a long time. It took until the 1960's for the equation to reappear in the numerical work of Kruskal and Zabusky [2] on nonlinear interactions in crystals and plasmas. What they discovered was essentially that suitable initial conditions could lead to solutions, which behave as a 'train of pulses' of the form (1.2) for asymptotic times (i.e., for $t \to \pm\infty$). Because the velocity of the pulses is proportional to their amplitudes, the ones with large amplitudes overtake the ones with small amplitudes and hence, for intermediate values of t, the pulses interact. After the interaction the individual pulses reappear without having lost their shape! These solutions are called N-soliton solutions, N referring to the number of pulses in the train.

Kruskal and Zabusky's work triggered a lot of further research in the field of nonlinear models. At present one knows that there are many other nonlinear partial differential equations (pdes), which allow soliton-like solutions. (The term soliton is not well-defined, but is generally used to refer to functions with the same qualitative behaviour as sketched above.) We mention some examples:

$$\begin{aligned} &\frac{\partial f}{\partial t} = i\frac{\partial^2 f}{\partial x^2} + 2|f|^2 f && \text{nonlinear Schrödinger equation,} \\ &\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} = \sin f && \text{sine-Gordon equation,} \\ &\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} = \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^4 f}{\partial x^4} && \text{Boussinesq-equation.} \end{aligned} \tag{1.3}$$

Notice that these are all pdes in one space variable x and one time variable t. There is also a soliton equation in two space variables x and y, and one time variable t, which can be considered as a two dimensional version of the KdV-equation:

$$\frac{3}{4}\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial t} - \frac{3}{2}f\frac{\partial f}{\partial x} - \frac{1}{4}\frac{\partial^3 f}{\partial x^3}\right]. \tag{1.4}$$

This equation is known as the Kadomtzev-Petviashvili equation.

Nowadays it is well known that infinite dimensional Lie algebras and the groups associated to them play an important, if not dominating, role in the theory of these soliton equations. The pioneering work in this direction was done in the early 80's by a group of Japanese mathematicians (see, e.g. [3,4]). The basic idea is, roughly speaking, that an infinite dimensional group acts as a 'symmetry group' on the space of solutions of a soliton equation, i.e., that it transfers solutions into each other. This means that, if one can identify the symmetry group for a particular soliton equation and its corresponding action on the solutions, it should be possible to construct solutions starting from a trivial one ($f=0$). We will explain this idea for the concrete example of the KP-equation (1.4).

Define the following formal differential operator:

$$X(u,v)= \tag{1.5}$$

$$\frac{1}{1-v/u}\exp\left[\sum_{k>0}(u^k-v^k)x_k\right]\exp\left[-\sum_{k>0}\frac{1}{k}(u^{-k}-v^{-k})\frac{\partial}{\partial x_k}\right]-\frac{1}{1-v/u}I.$$

Such an operator is to be understood as a power series in the formal variables u and v, the coefficients of which are differential operators in the variables $x_1,x_2,x_3,\ldots$. In physics it is called a vertex operator. In terms of these operators we can define the function

$$f_{a_1,a_2,\ldots,a_N,u_1,u_2,\ldots,u_N,v_1,v_2,\ldots,v_N}= \tag{1.6}$$

$$2\frac{\partial^2}{\partial x_1^2}\log\left[e^{a_1X(u_1,v_1)}e^{a_2X(u_2,v_2)}\cdots e^{a_NX(u_N,v_N)}.1\right],$$

where $a_1,a_2,\ldots,a_N,u_1,u_2,\ldots,u_N,v_1,v_2,\ldots,v_N\in\mathbb{C}$ are arbitrary parameters. Putting $x:=x_1, y:=x_2$, $t:=x_3$ and $x_4=x_5=\cdots=0$, we obtain the N-soliton solution of the KP-equation.

Of course one may wonder what the role of all other variables $x_4,x_5,\ldots$ is. The answer to this is, that the KP-equation is best seen as the first member of an infinite family of partial differential equations, the KP-hierarchy. The other members of the hierarchy are pdes in the remaining variables $x_4,x_5,\ldots$. In this context the function (1.6), which is a function of *all* variables $x_1,x_2,\ldots$, satisfies *all* equations of the KP-hierarchy. A similar remark holds for the other soliton equations (1.1) and (1.3).

It is worthwile to remark that the N-soliton solution of the KdV-hierarchy can be obtained from (1.6) by putting $v=-u$. From the expression (1.5) for the vertex operator one sees that the effect of this substitution is that all variables with even indices drop out. In particular, the solution (1.6) does not depend on y anymore. The reader should check that for $N=1$ one indeed finds the soliton solution (1.2) for the KdV (where $\alpha=(1/2u)\log a_1$).

Let us return to the N-soliton solution (1.6) of the KP-hierarchy. If we take $a_1=a_2=\cdots=a_N=0$, in other words, if we replace the exponentials by identity operators, we obtain the trivial solution $f=0$. So the N-soliton solution is indeed obtained from the action of a group on a trivial solution, as stated above. The group in question is the group of operators of the form

$$e^{a_1X(u_1,v_1)}e^{a_2X(u_2,v_2)}\cdots e^{a_NX(u_N,v_N)}; \tag{1.7}$$

$$a_1,a_2,\ldots,a_N,u_1,u_2,\ldots,u_N,v_1,v_2,\ldots,v_N\in\mathbb{C},N\in\mathbb{N}.$$

If we want to know more about this group, we should study the vertex operator $X(u,v)$. For this we write this operator as a formal power series in u and v

$$X(u,v) = \sum_{i,j\in\mathbf{Z}} X_{ij}u^i v^{-j}. \tag{1.8}$$

The homogeneous components X_{ij} of $X(u,v)$ in this power series are complicated differential operators, whose action is well defined on the space of polynomials in all variables $x_1,x_2,...$. But more is true; it can be shown that the collection of operators X_{ij} together with the identity operator generate a Lie algebra isomorphic to a central extension $\hat{gl}(\infty)$ of the infinite matrix algebra. From this observation one concludes that the symmetry group of the KP-hierarchy is a central extension of the group of invertible $\infty\times\infty$ matrices. (NB: the notion of a central extension will be explained below.)

These observations motivate the study of the infinite matrix algebra and its (projective) representations. In this paper, which is mainly expository, we present a survey of this theory, which should be accessible to non-specialists. The construction of the so-called 'semi-infinite wedge representation' in Sections 3 and 4 is in the spirit of the paper [5]. We thought it appropriate to motivate this construction with some classical facts about the 'wedge representations' of the finite dimensional matrix algebra $gl_n(\mathbf{C})$; these will be presented in Section 2.

Our main goal is to derive the formula (1.5) for the vertex operator and to explain why the homogeneous components of this operator really generate a Lie algebra isomorphic to a central extension of $gl(\infty)$. Our derivation is somewhat unconventional in the sense that it strongly emphasizes the role of the Virasoro algebra. This algebra, which is a central extension of the well-known algebra of infinitesimal conformal transformations of the complex plane, emerges in a completely natural manner when one studies the energy spectrum of an infinite collection of bosonic or fermionic oscillators. This will be explained in Sections 5 and 6. An excellent reference for the Virasoro algebra and its representations is the book [6]. We consider the fact that the bosonic and fermionic constructions of the Virasoro algebra coincide (Theorem 6.6) as the key result in the derivation of the expression for the vertex operator. This derivation can be found in Section 7.

Finally, in Section 8, we will give a brief account of our recent work [7] on multi-component fermionic constructions of the representations of $\hat{gl}(\infty)$. It is hoped that this work will shed some more light on multi-component KP-hierarchies, which are generalizations of the ordinary (1-component) KP-hierarchy.

2. The Lie algebra $gl_n(\mathbf{C})$

In order to motivate what is coming, we will first recall some basic facts about the finite dimensional representations of the Lie algebra $gl_n(\mathbf{C})$, the collection of all $n\times n$ matrices with complex entries. First some notation; let $\{e_i\}_{1\leq i\leq n}$ be the standard basis of $\mathbf{C}^n$ and let $E_{ij}, 1\leq i,\ j\leq n$ be the matrix with a 1 on the (i,j)-th entry and zeros elsewhere, i.e.,

$$E_{ij}e_k = \delta_{jk}e_i. \tag{2.1}$$

In terms of these matrices the Lie algebra structure of $gl_n(\mathbf{C})$ is given by the commutation relations

$$[E_{ij}, E_{kl}] = E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{jk}E_{il} - \delta_{il}E_{kj}. \tag{2.2}$$

By a representation of $gl_n(\mathbf{C})$ we mean a linear mapping $\pi : gl_n(\mathbf{C}) \rightarrow gl(V)$ of this Lie algebra to the endomorphisms of a vector space V such that the commutation relations (2.2) are respected, i.e.

$$[\pi(E_{ij}), \pi(E_{kl})] = \delta_{jk}\pi(E_{il}) - \delta_{il}\pi(E_{kj}). \tag{2.3}$$

The pair (V, π) is also called a module (over $gl_n(\mathbf{C})$). It is clear that the space $\mathbf{C}^n$ is a representation of $gl_n(\mathbf{C})$. It is called the defining representation or self representation of $gl_n(\mathbf{C})$. Although this representation is very simple, it has all the characteristics of a so-called highest weight representation.

Definition 2.1. Let λ be a linear mapping from the collection of diagonal matrices to $\mathbf{C}$. A representation $V(\lambda)$ of $gl_n(\mathbf{C})$ is called a highest weight representation, with highest weight λ if it satisfies the following properties:
(1) There exists a vector $v_\lambda \in V$, unique up to multiples, which is annihilated by all strictly upper triangular matrices in $gl_n(\mathbf{C})$.
(2) Any element of V can be obtained from the action of the lower triangular matrices in $gl_n(\mathbf{C})$ on this v_λ.
(3) The diagonal matrices act diagonally on v_λ; $\pi(D)(v_\lambda) = \langle \lambda, D \rangle v_\lambda$.
The vector v_λ is called a highest weight vector.

In the example of $V(\lambda_1) = \mathbf{C}^n$ we simply take $v_{\lambda_1} = e_1$ and $< \lambda_1, \text{diag}(d_1, \ldots, d_n) > := d_1$.

There is one more important property of the defining representation; the only subspaces $U \subset \mathbf{C}^n$ which are invariant under the action of $gl_n(\mathbf{C})$ are $U = 0$ and $U = \mathbf{C}^n$. In other words the defining representation is irreducible.

If V_1 and V_2 are any two representations of $gl_n(\mathbf{C})$, one defines the tensor product representation as the space $V_1 \otimes V_2$ with the action:

$$\pi(A)(v_1 \otimes v_2) := \pi_1(A)(v_1) \otimes v_2 + v_1 \otimes \pi_2(A)(v_2) \tag{2.4}$$
$$\forall A \in gl_n(\mathbf{C}),\ v_1 \in V_1,\ v_2 \in V_2.$$

Let us take $V_1 = V_2 = \mathbf{C}^n$. The tensor product $\mathbf{C}^n \otimes \mathbf{C}^n$ is clearly not irreducible; it is the direct sum of the spaces of symmetric and antisymmetric tensors, which are both invariant under the action (2.4). Let us denote the space of antisymmetric tensors by $\wedge^2 \mathbf{C}^n$. It can be shown that this space is again an irreducible highest weight representation of $gl_n(\mathbf{C})$. Its highest weight vector is $e_2 \wedge e_1$ and its highest weight is given by $< \lambda_2, \text{diag}(d_1, \ldots, d_n) > := d_1 + d_2$. Continuing in this manner, we are led to the so-called k-th wedge space $\wedge^k \mathbf{C}^n$ as the k-fold exterior product of the vector space $\mathbf{C}^n$ with $gl_n(\mathbf{C})$-action:

$$\pi_k(A)(v_1 \wedge v_2 \wedge \cdots \wedge v_k) := (Av_1) \wedge v_2 \wedge \cdots \wedge v_k + v_1 \wedge (Av_2) \wedge \cdots \wedge v_k + \cdots + v_1 \wedge v_2 \wedge \cdots \wedge (Av_k) \quad \forall A \in gl_n(\mathbb{C}),\ v_1, v_2, \ldots, v_k \in \mathbb{C}^n. \tag{2.5}$$

We can now formulate the following

Lemma 2.2. *The spaces* $V_k := \wedge^k \mathbb{C}^n, 1 \leq k \leq n$ *are irreducible highest weight modules over* $gl_n(\mathbb{C})$ *with highest weight vector* $e_k \wedge e_{k-1} \wedge \cdots \wedge e_1$ *and highest weight given by* $< \lambda_k, diag\,(d_1, \ldots, d_n)> := d_1 + d_2 + \cdots + d_k$.

3. Lie algebras of infinite matrices

In this section we will show how to generalize the notions from the previous section to Lie algebras of $\infty \times \infty$ matrices. The most straightforward manner to do this, is to consider the collection $gl(+\infty)$ consisting of all matrices $(a_{ij})_{i,j \in \mathbf{N}}$ such that all but a finite number of entries a_{ij} are zero. It is easy to verify that the matrix product and hence also the commutator is well defined in this algebra. We can also introduce the vector space $\mathbb{C}^{+\infty}$ as the set of all column vectors $(x_i)_{i \in \mathbf{N}}$ such that all but a finite number of x_i's are zero. The algebra $gl(+\infty)$ acts in the obvious manner on $\mathbb{C}^{+\infty}$ and we can again call this space the defining representation of $gl(+\infty)$. Moreover, $\mathbb{C}^{+\infty}$ is again an irreducible highest weight module and the same holds for the k-fold exterior product $\wedge^k \mathbb{C}^{+\infty}$. We conclude that this generalization is completely trivial: nothing interesting happens.

The situation changes dramatically if we consider instead of $gl(+\infty)$ the Lie algebra $gl(\infty)$, defined as the collection of all matrices $(a_{ij})_{i,j \in \mathbf{Z}}$ such that all but a finite number of entries a_{ij} are zero. The defining representation of this algebra is of course the space $\mathbb{C}^\infty$ consisting of all vectors $(x_i)_{i \in \mathbf{Z}}$ such that almost all x_i are zero. Now consider the collection of all strictly upper triangular matrices in $gl(\infty)$. Because the matrices in $gl(\infty)$ are infinite in both the 'positive' and in the 'negative' direction, the strictly upper triangular matrices in $gl(\infty)$ do not have a common column of zeros. This means that there is no vector in $\mathbb{C}^\infty$ which is annihilated by all such matrices. Consequently, the defining representation is not a highest weight representation for $gl(\infty)$ and neither is any of its finite exterior products $\wedge^k \mathbb{C}^\infty$.

The question arises if there are any highest weight representations of $gl(\infty)$ at all. The answer to this question is yes; to see this we introduce, following Kac and Peterson [5], the semi-infinite wedge space $\wedge^\infty \mathbb{C}^\infty$ as the vector space consisting of all finite linear combinations of semi-infinite exterior products of the basis elements e_i of $\mathbb{C}^\infty$ of the form:

$$e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \tag{3.1}$$

such that $i_0 > i_1 > i_2 > \cdots$ and such that $i_{l+1} = i_l - 1$ for $l \gg 0$. On this space $gl(\infty)$ acts as usual; denoting the action by τ, we can write

$$\tau(A)(e_{i_0}\wedge e_{i_1}\wedge e_{i_2}\wedge\cdots) = (Ae_{i_0})\wedge e_{i_1}\wedge e_{i_2}\wedge\cdots + e_{i_0}\wedge(Ae_{i_1})\wedge e_{i_2}\wedge\cdots + e_{i_0}\wedge e_{i_1}\wedge(Ae_{i_2})\wedge\cdots+\cdots \quad \forall A\in gl(\infty). \tag{3.2}$$

We can distinguish the basis elements (3.1) by their behaviour at large l; we will say that an element of the form (3.1) has charge k if $i_l = k-l$ for all $l \gg 0$. For instance the vector

$$|k\rangle \equiv v_k := e_k\wedge e_{k-1}\wedge e_{k-2}\wedge\cdots \tag{3.3}$$

has charge k. In this case we will refer to v_k as the k-th vacuum. The vector space of all vectors of charge k is denoted by $\wedge_k^\infty\mathbb{C}^\infty$ and we clearly have a decomposition of the full semi-infinite wedge space in sectors of fixed charge

$$\wedge^\infty\mathbb{C}^\infty = \bigoplus_{k\in\mathbb{Z}}\wedge_k^\infty\mathbb{C}^\infty. \tag{3.4}$$

In fact this decomposition is a decomposition in submodules for the action τ of the algebra $gl(\infty)$; the action of an element E_{ij} on a wedge $e_{i_0}\wedge e_{i_1}\wedge e_{i_2}\wedge\cdots$ replaces the vector e_j in this wedge—if present—by e_i and this does not effect the charge of the wedge. Even more is true: the submodule $\wedge_k^\infty\mathbb{C}^\infty$ is a highest weight module for the algebra $gl(\infty)$. We have for $j>i$

$$\tau(E_{ij})(v_k) = \delta_{jk}e_i\wedge e_{k-1}\wedge e_{k-2}\wedge\cdots + \delta_{j,k-1}e_k\wedge e_i\wedge e_{k-2}\wedge\cdots+\cdots = 0. \tag{3.5}$$

In other words: the k-th vacuum is annihilated by all strictly upper triangular matrices. Moreover, it is easily seen that any wedge in $\wedge_k^\infty\mathbb{C}^\infty$ of the form (3.1) can be obtained by successive application of the action of strictly lower triangular matrices E_{ij} on the vacuum

$$\tau(E_{i_0,k})\tau(E_{i_1,k-1})\tau(E_{i_2,k-2})\cdots(v_k) = e_{i_0}\wedge e_{i_1}\wedge e_{i_2}\wedge\cdots \tag{3.6}$$

Finally, the diagonal matrices act by multiplication with a scalar on v_k

$$\tau(E_{ii})(v_k) = \langle\theta_k, E_{ii}\rangle v_k, \tag{3.7}$$

where the linear mapping $\theta_k : \bigoplus_{i\in\mathbb{Z}}\mathbb{C}E_{ii}\to\mathbb{C}$ is defined by

$$\langle\theta_k, E_{ii}\rangle := \begin{cases} 0 & \text{if } i>k, \\ 1 & \text{if } i\leqslant k. \end{cases} \tag{3.8}$$

This means that $\wedge_k^\infty\mathbb{C}^\infty$ is a highest weight module for the Lie algebra $gl(\infty)$ with highest weight θ_k. It is not too difficult to prove that this module is in fact irreducible. In the sequel we will denote the restriction of the representation τ to this module by τ_k.

Next we introduce elementary 'creation' and 'annihilation' operators on the semi-infinite wedge space $\wedge^\infty\mathbb{C}^\infty$; for every $i\in\mathbb{Z}$ we define linear operators $\psi(i)$ and $\psi^*(i)$ on the semi-infinite wedge space by their action on basis vectors:

$$\psi(i)(e_{i_0}\wedge e_{i_1}\wedge e_{i_2}\wedge\cdots):=e_i\wedge e_{i_0}\wedge e_{i_1}\wedge e_{i_2}\wedge\cdots,$$

$$\psi^*(i)(e_{i_0}\wedge e_{i_1}\wedge e_{i_2}\wedge\cdots):=\sum_{k=0}^{\infty}(-)^k\delta_{i,i_k}e_{i_0}\wedge e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge\hat{e}_{i_k}\wedge\cdots, \tag{3.9}$$

where the notation $\hat{e}i_k$ means that the vector e_{i_k} is deleted. The restrictions of these operators to a fixed charge sector raise and lower the charge:

$$\psi(i):\wedge_k^\infty\mathbf{C}^\infty\to\wedge_{k+1}^\infty\mathbf{C}^\infty,$$
$$\psi^*(i):\wedge_k^\infty\mathbf{C}^\infty\to\wedge_{k-1}^\infty\mathbf{C}^\infty. \tag{3.10}$$

Furthermore, these operators satisfy anticommutation relations:

$$\{\psi(i),\psi(j)\}=0=\{\psi^*(i),\psi^*(j)\},\quad \{\psi(i),\psi^*(j)\}=\delta_{ij}. \tag{3.11}$$

The importance of these operators lies in the fact that the action of the elements E_{ij} can be expressed as a product of these operators; one clearly has

$$\tau(E_{ij})=\psi(i)\psi^*(j). \tag{3.12}$$

Note that any element of the semi-infinite wedge space $\wedge^\infty\mathbf{C}^\infty$ can be written as a finite linear combination of elements of the form

$$\psi(i_0)\psi(i_1)\cdots\psi(i_k)\psi^*(j_0)\psi^*(j_1)\cdots\psi^*(j_l)(v_0),$$
$$i_0>i_1>\cdots>i_k>0\geqslant j_0>j_1>\cdots>j_l. \tag{3.13}$$

This means that we could also have constructed $\wedge^\infty\mathbf{C}^\infty$ in a superficially different manner, which is more familiar to physicists. Let Cl be the Clifford algebra on generators $\psi(i)$, $\psi^*(i)$, $i\in\mathbf{Z}$ with relations (3.11). Define the so-called fermionic Fock-space F as the unique irreducible Cl-module, which admits a vacuum vector $|0>$ such that

$$\psi(i)|0>=0\quad\forall i\leqslant 0,$$
$$\psi^*(i)|0>=0\quad\forall i>0. \tag{3.14}$$

With the identification $|0>=v_0$ we have $F=\wedge^\infty\mathbf{C}^\infty$. The fermionic Fock space F is also called the spin representation of Cl.

In physics one introduces a charge operator q by:

$$q|0>=0,\quad [q,\psi(k)]=\psi(k),\quad [q,\psi^*(k)]=-\psi^*(k). \tag{3.15}$$

It is clear that this is the same notion of charge as the one introduced above.

4. Completions, central extensions and normal ordering

For many purposes the Lie algebra $gl(\infty)$ is too small. Therefore, one usually considers a completion of this algebra.

DEFINITION 4.1. The Lie algebra $\overline{gl(\infty)}$ is the collection of all $\infty\times\infty$ matrices of finite width around the main diagonal, that is,

$$\overline{gl(\infty)}:=\{\sum_{i,j\in\mathbf{Z}}g_{ij}E_{ij}|g_{ij}\in\mathbf{C}\wedge g_{ij}=0\text{ if }|i-j|\gg 0\}. \tag{4.1}$$

If one tries to extend the representation τ to the algebra $\overline{gl(\infty)} \supset gl(\infty)$ by linearity, one immediately runs into problems. These problems originate from the fact that the action of diagonal matrices in $\overline{gl(\infty)}$ is not well defined if we simply extend τ by linearity; if we consider, e.g., the action of the identity matrix $\Sigma_{i\in\mathbf{Z}} E_{ii} \in \overline{gl(\infty)}$ on the vacuum vector v_0, the result would be the vacuum vector multiplied by:

$$\langle \theta_0, \sum_{i\leqslant 0} E_{ii} \rangle = \infty. \tag{4.2}$$

It is well known how to repair this situation; one simply replaces the representation τ by the assignment π defined by:

$$\pi(E_{ij}) := \tau(E_{ij}) - \delta_{ij} \langle \theta_0, E_{ii} \rangle I. \tag{4.3}$$

The assignment π can be extended to diagonal matrices in $\overline{gl(\infty)}$; for the action of the identity matrix on the k-th vacuum we find:

$$\pi \left[\sum_{i\in\mathbf{Z}} E_{ii} \right] (v_k) = \sum_{i\in\mathbf{Z}} \langle \theta_k - \theta_0, E_{ii} \rangle v_k = k v_k. \tag{4.4}$$

It can also be proved that π can be extended to strictly upper and strictly lower triangular matrices in $\overline{gl(\infty)}$.

Of course π is not a representation of $gl(\infty)$ any more, that is,

$$\begin{aligned}[] [\pi(E_{ij}), \pi(E_{kl})] &= [\tau(E_{ij}), \tau(E_{kl})] \\ &= \delta_{jk}\tau(E_{il}) - \delta_{il}\tau(E_{kj}) \\ &= \delta_{jk}\pi(E_{il}) - \delta_{il}\pi(E_{kj}) + \delta_{il}\delta_{jk} \langle \theta_0, E_{ii} - E_{jj} \rangle. \end{aligned} \tag{4.5}$$

Because of the extra term $\delta_{il}\delta_{jk}\langle \theta_0, E_{ii} - E_{jj}\rangle$ in the right-hand side of (4.5), π is called a projective representation of $gl(\infty)$. Another way to formulate this is to introduce a new algebra $\hat{gl}(\infty)$ as the vector space $gl(\infty) \oplus \mathbf{C}c$ with commutation relations

$$[A + \alpha c, B + \beta c] := [A,B] + \mu(A,B)c, \quad \forall A,B \in gl(\infty), \ \alpha,\beta \in \mathbf{C}. \tag{4.6}$$

The commutator $[A,B]$ in the right-hand side of this formula is the ordinary commutator in $gl(\infty)$ and the bilinear mapping $\mu : gl(\infty) \times gl(\infty) \to \mathbf{C}$ is defined by:

$$\mu(E_{ij}, E_{kl}) := \delta_{il}\delta_{jk} \langle \theta_0, E_{ii} - E_{jj} \rangle. \tag{4.7}$$

This mapping is called a two cocycle on $gl(\infty)$. The algebra $\hat{gl}(\infty)$ is called a central extension of $gl(\infty)$ and the assignment π can be considered as a genuine representation of this central extension, in which the central element is represented by the identity operator. The definition 4.1 of the algebra $\overline{gl(\infty)}$ guarantees that this two cocycle can be extended to this algebra by linearity. The Lie algebra $\overline{gl(\infty)} \oplus \mathbf{C}c$ with commutation relations defined by this two cocycle is called A_∞.

Notice that we have the following expression for the two cocycle μ:

$$\mu(A,B) = \text{trace}(H[A,B]), \quad \forall A,B \in \overline{gl(\infty)}, \tag{4.8}$$

where $H := \Sigma_{i \leq 0} E_{ii}$. A word of caution is justified here; formula (4.8) shows that the value of the two cocycle on two elements A and B of $\overline{gl(\infty)}$ depends only on the commutator $[A,B]$ and one is tempted to modify the operator $\pi(a)$ by trace$(HA)I$ and conclude that the central extension is trivial. Notice however that the one cycle trace(HA) is only well defined for $A \in gl(\infty)$. On this algebra we have $\pi(a) + \text{trace}\,(HA)I = \tau(A)$, whence the restriction of the central extension to $gl(\infty) \subset \overline{gl(\infty)}$ is trivial. On the completion $\overline{gl(\infty)}$ this cannot be done and the central extension is nontrivial.

In physics the procedure above is usually formulated in terms of a normal ordering prescription on the fermionic creation and annihilation operators; one writes:

$$\pi(E_{ij}) = :\psi(i)\psi^*(j):, \tag{4.9}$$

where the normal ordered product $:\psi(i)\psi^*(j):$ is defined by:

$$:\psi(i)\psi^*(j): := \psi(i)\psi^*(j) - \delta_{ij}\langle\theta_0, E_{ii}\rangle = \begin{cases} \psi(i)\psi^*(j) & \text{if } i > 0, \\ -\psi^*(j)\psi(i) & \text{if } i \leq 0. \end{cases} \tag{4.10}$$

5. The energy spectrum of $\wedge^\infty \mathbf{C}^\infty$ and the Virasoro algebra

In physics the semi-infinite wedge space $\wedge^\infty \mathbf{C}^\infty$ is interpreted as the Hilbert space for a system of fermionic oscillators. In this context one introduces the following 'energy' operator on $\wedge^\infty \mathbf{C}^\infty$:

$$H_0 := \sum_{k \in \mathbf{Z}} k :\psi(k)\psi^*(k):, \tag{5.1}$$

or in terms of the algebra A_∞:

$$H_0 = \pi\left[\sum_{k \in \mathbf{Z}} k E_{kk}\right]. \tag{5.2}$$

With the normal ordering definition (4.11) and the anticommutation relations (3.11) one easily derives:

$$\begin{aligned} &H_0(v_0) = 0, \\ &[H_0, \psi(k)] = k\psi(k), \\ &[H_0, \psi^*(k)] = -k\psi^*(k). \end{aligned} \tag{5.3}$$

From these relations it is clear that the creation of a fermion from the vacuum v_0 with the operator $\psi(k)$, $k > 0$ requires an energy k. Similarly, the annihilation of a fermion with the operator $\psi(k)$, $k \leq 0$ yields an energy $-k$. It is also clear that H_0 is diagonalizable with eigenvalues in $\mathbf{Z}_{\geq 0}$; its eigenvectors are precisely the elements $\psi(i_0)\psi(i_1) \cdots \psi(i_k)\psi^*(j_0)\psi^*(j_1) \cdots \psi^*(j_l)(v_0)$, $i_0 > i_1 > \cdots > i_k > 0 \geq j_0 > j_1 > \cdots > j_l$ from (3.13). The energy of such a vector is $i_0 + i_1 + \cdots + i_k - (j_0 + j_1 + \cdots + j_l)$. Because the minimal eigenvalue of H_0 is zero, one speaks of a positive energy representation of A_∞.

Let z be a formal parameter and introduce the following formal fermionic fields:

$$\psi(z) := \sum_{k\in\mathbf{Z}} \psi(k) z^k,$$
$$\psi^*(z) := \sum_{k\in\mathbf{Z}} \psi^*(k) z^{-k}. \tag{5.4}$$

In terms of these fields the commutation relations (5.3) can be expressed as:

$$[H_0,\psi(z)] = z\frac{d}{dz}\psi(z),$$
$$[H_0,\psi^*(z)] = z\frac{d}{dz}\psi^*(z). \tag{5.5}$$

In view of this formula it is natural to look for operators H_n, $n\in\mathbf{Z}$ on $\wedge^\infty\mathbf{C}^\infty$ whose adjoint action on $\psi(z)$ is given by $z^{n+1}(d/dz)$. A small calculation yields that the operators $\Sigma_{k\in\mathbf{Z}}(k-n):\psi(k-n)\psi^*(k):$ satisfy this requirement. The adjoint action of these operators on the conjugate field is given by $z^{n+1}(d/dz)+nz^n$. For reasons to be explained in the sequel we will work with slightly different operators, viz

$$H_n := \sum_{k\in\mathbf{Z}}(k-n\beta):\psi(k-n)\psi^*(k): = \pi\left[\sum_{k\in\mathbf{Z}}(k-n\beta)E_{k-n,k}\right], \tag{5.6}$$

where the value of $\beta\in\mathbf{R}$ will be chosen below. The commutation relations of these operators with the fermionic fields are:

$$[H_n,\psi(z)] = z^n\left[z\frac{d}{dz}+n(1-\beta)\right]\psi(z),$$
$$[H_n,\psi^*(z)] = z^n\left[z\frac{d}{dz}+n\beta\right]\psi^*(z). \tag{5.7}$$

Next we compute the commutation relations between the H_n's;

LEMMA 5.1 (cf. [6]).

$$[H_m,H_n]= \tag{5.8}$$
$$(m-n)\left[H_{m+n}+\frac{1}{2}\delta_{m+n,0}\beta(1-\beta)\right]+\frac{1}{12}(m^3-m)\delta_{m+n,0}c_\beta I,$$

where

$$c_\beta = -12\beta^2+12\beta-2. \tag{5.9}$$

PROOF. With the expression (5.6) of H_n in terms of the representation π of the algebra A_∞ we can write for the commutator $[H_m,H_n]$:

$$\pi\left[\sum_{k,l\in\mathbf{Z}}(k-m\beta)(l-n\beta)[E_{k-m,k},E_{l-n,l}]\right]+\sum_{k,l\in\mathbf{Z}}(k-m\beta)(l-n\beta)\mu(E_{k-m,k},E_{l-n,l}).$$

The first term in this expression can be computed with the commutation relations in $gl(\infty)$; one finds

$$(m-n)\pi\left[\sum_{k\in\mathbf{Z}}(k-(m+n)\beta)E_{k-(m+n),k}\right]=(m-n)H_{m+n}.$$

The second term becomes:

$$\sum_{k,l\in\mathbf{Z}}(k-m\beta)(l-n\beta)\delta_{l,k+n}\delta_{k,l+m}\mu(E_{k-m,k},E_{k,k-m})=$$

$$\delta_{m+n,0}\sum_{k\in\mathbf{Z}}(k-m\beta)(k-m(1-\beta))\mu(E_{k-m,k},E_{k,k-m}).$$

For $m>0$ we get

$$\delta_{m+n,0}\sum_{k=1}^{m}(k-m\beta)(k-m(1-\beta))=\delta_{m+n,0}\sum_{k=1}^{m}(k^2-mk+\beta(1-\beta)m^2)$$

$$=\delta_{m+n,0}\frac{1}{12}(m^3-m)c_\beta+\delta_{m+n,0}m\beta(1-\beta).$$

The case $m<0$ is similar. Combining the two results, we have proved the lemma. □

The lemma shows that the 'shifted energy' operators $\hat{H}_n:=H_n+½\delta_{n0}\beta(1-\beta)I$ satisfy the commutation relations

$$[\hat{H}_m,\hat{H}_n]=(m-n)\hat{H}_{m+n}+\frac{1}{12}\delta_{m+n,0}(m^3-m)c_\beta I. \tag{5.10}$$

The algebra on a basis $\{D_n\}_{n\in\mathbf{Z}}\cup\{c_{vir}\}$ with commutation relations

$$[D_m,D_n]=(m-n)D_{m+n}+\frac{1}{12}(m^3-m)c_{vir};\quad [D_n,c_{vir}]=0, \tag{5.11}$$

is called the Virasoro algebra. It is a one-dimensional central extension of the conformal algebra, i.e., the algebra spanned by elements d_n, $n\in\mathbf{Z}$ with commutation relations $[d_m,d_n]=(m-n)d_{m+n}$. From (5.10) we see that the assignment $D_n\mapsto\hat{H}_n$ is a representation of the Virasoro algebra in which the central element c_{vir} is represented by $c_\beta I$.

In the next section it will be crucial that $c_\beta=1$ or, equivalently, that $\beta=½$. In this case one has (see (5.7)):

$$[\hat{H}_n,\psi(z)]=z^n\left[z\frac{d}{dz}+\frac{1}{2}n\right]\psi(z),$$

$$[\hat{H}_n,\psi^*(z)]=z^n\left[z\frac{d}{dz}+\frac{1}{2}n\right]\psi^*(z). \tag{5.12}$$

These relations express that the fermionic fields $\psi(z)$ and $\psi^*(z)$ have conformal weights ½ (see, e.g., [8]).

Right now we can also give another explanation for choosing $\beta=½$. For this we recall that the semi-infinite wedge space can be equipped with an inner product, which is uniquely determined by the requirements

$$\text{a) } \psi(k)^\dagger = \psi^*(k), \quad \text{b) } (v_0, v_0) = 1. \tag{5.13}$$

The commutation relations of the conformal algebra can of course be realized by choosing $d_n := -\lambda^{n+1}(d/d\lambda)$. If we read λ as $e^{i\theta}$ this becomes $d_n := ie^{in\theta}(d/d\theta)$, so it is natural to demand that any Hilbert space representation of this algebra satisfies $d_n^\dagger = d_{-n}$. Motivated by this we also require for the Virasoro operators $\hat{H}_n$:

$$\hat{H}_n^\dagger = \hat{H}_{-n}. \tag{5.14}$$

A representation of the Virasoro algebra satisfying this requirement will be called unitary. It is clear that with the inner product determined by (5.13), this can only be achieved if $\beta = ½$.

6. Bosons

Here we introduce the operators $\alpha(k)$, $k \in \mathbf{Z}$ by:

$$\alpha(k) := \sum_{j \in \mathbf{Z}} :\psi(j)\psi^*(j+k): = \pi\left[\sum_{j \in \mathbf{Z}} E_{j,j+k}\right]. \tag{6.1}$$

With the inner product on $\wedge^\infty \mathbf{C}^\infty$ determined by (5.13) one easily verifies:

$$\alpha(k)^\dagger = \alpha(-k). \tag{6.2}$$

Notice that the Hermitian operator $\alpha(0)$ is simply the charge operator (see (3.15) and (4.4)) on the fermionic Fock space.

The commutation relations between the $\alpha(k)$'s are given by the following lemma.

Lemma 6.1.

$$[\alpha(k), \alpha(j)] = k\delta_{k+j,0} I. \tag{6.3}$$

Proof. The operator $\alpha(k)$ corresponds to the k-th diagonal in the algebra $gl(\infty)$. Since the k-th and the j-th diagonal commute in $gl(\infty)$, we only get a contribution from the two cocycle μ; for $k > 0$ this becomes:

$$\mu\left(\sum_{m \in \mathbf{Z}} E_{m,m+k}, \sum_{n \in \mathbf{Z}} E_{n,n+j}\right) = \sum_{m,n \in \mathbf{Z}} \delta_{n,m+k}\delta_{m,n+j}\mu(E_{m,m+k}, E_{m+k,m})$$

$$= \delta_{k+j,0} \sum_{m=-k+1}^{0} \mu(E_{m,m+k}, E_{m+k,m})$$

$$= k\delta_{k+j,0}.$$

The case $k < 0$ is similar. □

The relations (6.3) are the quantum mechanical commutation relations for a system of bosonic oscillators. If one leaves out the 'zero mode' $\alpha(0)$, which is constant on a fixed charge sector, one can define for $k>0$ $p_k := \alpha(k)$, $q(k) := \frac{1}{k}\alpha(-k)$. In terms of these operators we can write:

$$[p_k, q_j] = \delta_{kj} I. \tag{6.4}$$

These relations are known as the Heisenberg commutation relations in physics. Notice that they can easily be realized on the ring of polynomials in all variables $x_1, x_2, x_3, \ldots$, by the assignment $p_k \mapsto \partial/\partial x_k$, $q_k \mapsto x_k$.

There is a standard recipe to construct a representation of the Virasoro algebra from a system of bosonic oscillators.

LEMMA 6.2. *Let* $\{a_i\}_{i\in\mathbf{Z}}$ *be a collection of operators on a vector space* V, *such that* $[a_i, a_j] = i\delta_{i+j,0} I$ *and* $a_i(v) = 0, \forall v \in V$ *and* $i \gg 0$. *Define normal ordering by* $:a_i a_j: = a_i a_j$ *if* $i \leqslant j$ *and* $:a_i a_j: = a_j a_i$ *if* $i > j$. *Then the operators* L_k, *defined by*

$$L_k := \frac{1}{2}\sum_{j\in\mathbf{Z}} :a_{-j}a_{j+k}:, \tag{6.5}$$

satisfy the commutation relations for the Virasoro algebra:

$$[L_k, L_j] = (k-j)L_{k+j} + \frac{1}{12}(k^3 - k)\delta_{k+j,0} I. \tag{6.6}$$

Moreover, we have:

$$[L_k, a_i] = -i a_{i+k}. \tag{6.7}$$

PROOF. See, e.g., [6]. □

In our situation we take $V = \wedge^\infty \mathbf{C}^\infty$ and $a_i = \alpha(i)$. In this case the relation (6.2) guarantees that we have again a unitary representation of the Virasoro algebra;

$$L_n^\dagger = L_{-n} \quad \forall n \in \mathbf{Z}. \tag{6.8}$$

The condition that the $\alpha(k)$'s annihilate an arbitrary fixed vector for k large enough is easily verified with the definition (6.1). Notice that we have introduced a normal ordering on the bosons, which, by abuse of notation, is again denoted by : :.

We now have two unitary representations of the Virasoro algebra with $c_{vir} \mapsto I$; in the previous section we have introduced the operators $\hat{H}_n$ in terms of normal ordered products of the fermionic oscillators $\psi(k)$ and $\psi^*(k)$. Here we have defined operators L_n in terms of normal ordered products of the bosonic oscillators $\alpha(k)$, which in turn can be written as a normal ordered product of fermions. In particular the $\hat{H}_n$'s are quadratic and the L_n's are quartic in the fermions and it is a priori not at all clear that these operators are related.

Let us further investigate this situation by computing the commutation relations of the $\hat{H}_k$'s with the oscillators $\alpha(i)$ (the commutators between L_k's and $\alpha(i)$'s are given by (6.7)).

LEMMA 6.3.

$$[\hat{H}_j,\alpha(i)] = -i\left[\alpha(i+j) + \frac{1}{2}\delta_{i+j,0}\right]. \tag{6.9}$$

PROOF.

$$\begin{aligned}
[\hat{H}_j,\alpha(i)] &= [\pi\left[\sum_{k\in\mathbf{Z}}(k-\frac{1}{2}j)E_{k-j,k}\right], \pi\left[\sum_{l\in\mathbf{Z}}E_{l,l+i}\right]] \\
&= \pi\left[\sum_{k,l\in\mathbf{Z}}(k-\frac{1}{2}j)[E_{k-j,k},E_{l,l+i}]\right] + \sum_{k,l\in\mathbf{Z}}(k-\frac{1}{2}j)\mu(E_{k-j,k},E_{l,l+i}) \\
&= \pi\left[\sum_{k\in\mathbf{Z}}(k-\frac{1}{2}j)E_{k-j,k+i} - (k-\frac{1}{2}j)E_{k-i-j,k}\right] \\
&\quad + \sum_{k,l\in\mathbf{Z}}(k-\frac{1}{2}j)\delta_{kl}\delta_{k-j,l+i}\mu(E_{k-j,k},E_{k,k-j}) \\
&= -i\alpha(i+j) + \delta_{i+j,0}\sum_{k\in\mathbf{Z}}(k-\frac{1}{2}j)\mu(E_{k-j,k},E_{k,k-j}).
\end{aligned}$$

For $j>0$ the central term becomes $\delta_{i+j,0}\Sigma_{k=1}^{j}(k-\frac{1}{2}j) = -\frac{1}{2}i\delta_{i+j,0}$. The case $j<0$ is similar. □

The lemma suggests of course to replace $\alpha(k)$ by $\hat{\alpha}(k) := \alpha(k) + \frac{1}{2}\delta_{k0}$. It is clear that the commutation relations (6.3) of the oscillators are not affected by this replacement. We can now use Lemma 6.2 again and construct Virasoro operators $\hat{L}_k$ in terms of the $\hat{\alpha}(k)$'s. Let us summarize what we have achieved sofar in a lemma.

LEMMA 6.4. *Define*

$$\hat{\alpha}(k) := \sum_{j\in\mathbf{Z}} :\psi(j)\psi^*(j+k): + \frac{1}{2}\delta_{k0}I, \tag{6.10}$$

then the $\hat{\alpha}(k)$*'s satisfy*

$$[\hat{\alpha}(k),\hat{\alpha}(j)] = k\delta_{k+j,0}. \tag{6.11}$$

Let

$$\begin{aligned}
\hat{L}_k &:= \frac{1}{2}\sum_{j\in\mathbf{Z}} :\hat{\alpha}(-j)\hat{\alpha}(j+k):, \\
\hat{H}_k &:= \sum_{j\in\mathbf{Z}}(j-\frac{1}{2}k) :\psi(j-k)\psi^*(j): + \frac{1}{8}\delta_{k0}.
\end{aligned} \tag{6.12}$$

then both the $\hat{L}_k$'s and the $\hat{H}_k$'s define a $c_{vir} \mapsto I$ representation of the Virasoro algebra. Moreover, we have:

$$[\hat{L}_j, \hat{\alpha}(i)] = -i\hat{\alpha}(i+j) = [\hat{H}_j, \hat{\alpha}(i)]. \tag{6.13}$$

The next step is to consider the difference $\Delta_k := \hat{H}_k - \hat{L}_k$. The following lemma states some crucial properties of Δ_k.

LEMMA 6.5.

$$\begin{aligned} &\text{a)}\ [\Delta_k, \Delta_j] = (k-j)\Delta_{k+j}, \\ &\text{b)}\ \Delta_k^{\dagger} = \Delta_{-k}, \\ &\text{c)}\ \Delta_k |j> = 0, \quad \forall k > 0 \ \forall j, \\ &\text{d)}\ \Delta_0 |j> = \lambda_j |j> \quad \lambda_j \in \mathbb{R}\ \forall j. \end{aligned} \tag{6.14}$$

PROOF. Relations b), c) and d) are immediate from the corresponding properties of $\hat{H}_k$ and $\hat{L}_k$. To prove a) we remark that the operators Δ_k commute with all oscillators $\hat{\alpha}(j)$;

$$[\Delta_k, \hat{\alpha}(j)] = [\hat{H}_k, \hat{\alpha}(j)] - [\hat{L}_k, \alpha(j)] = 0.$$

Because $\hat{L}_k$ is defined entirely in terms of the $\hat{\alpha}(i)$'s, we conclude:

$$[\Delta_k, \hat{L}_j] = 0 \quad \forall k, j.$$

Hence:

$$\begin{aligned} [\Delta_k, \Delta_j] &= [\Delta_k, \hat{H}_j - \hat{L}_j] \\ &= [\Delta_k, \hat{H}_j] \\ &= [\hat{H}_k, \hat{H}_j] - [\hat{L}_k, \hat{L}_j] - [\hat{L}_k, \Delta_j] \\ &= (k-j)\hat{H}_{k+j} - (k-j)\hat{L}_{k+j} \\ &= (k-j)\Delta_{k+j}. \end{aligned}$$

□

Notice that the proof of both a) and b) requires that $\beta = ½$.

The lemma means essentially that the operators Δ_k provide a unitary representation of the conformal algebra on the semi-infinite wedge space. It can be shown (see, e.g., [7]) that this representation must be trivial, i.e. $\Delta_k = 0\ \forall k$. Hence, we have the following theorem, which is really the crucial result in boson-fermion correspondence and the representation theory of A_∞.

THEOREM 6.6.

$$\hat{H}_k = \hat{L}_k \quad \forall k. \tag{6.15}$$

Notice that with the bosonic field defined by

$$\alpha(z) := :\psi(z)\psi^*(z): = \sum_{k \in \mathbf{Z}} \alpha(k) z^{-k}. \tag{6.16}$$

The theorem can be reformulated in a way which is more familiar to physicists;

$$\frac{1}{2}:\left[z\frac{d}{dz}\psi(z)\right]\psi^{*}(z):-\frac{1}{2}:\psi(z)\left[z\frac{d}{dz}\psi^{*}(z)\right]:=\frac{1}{2}:\alpha(z)^{2}:+\frac{1}{2}\alpha(z). \tag{6.17}$$

7. Vertex operators

Let u and v be formal parameters and introduce the vertex operator for the algebra A_{∞} as the following formal operator:

$$X(u,v):=\sum_{i,j\in\mathbf{Z}}\pi(E_{ij})u^{i}v^{-j}. \tag{7.1}$$

One should think of $X(u,v)$ as a formal operator valued power series, which generates the projective action of the Lie algebra $gl(\infty)$ on the semi-infinite wedge space; extracting the coefficient of $u^{i}v^{-j}$ yields the operator $\pi(E_{ij})$. In this section we will derive a well-known expression for $X(u,v)$ in terms of the bosonic oscillators.

Notice that (7.1) is equivalent to

$$X(u,v)=\sum_{i,j\in\mathbf{Z}}:\psi(i)\psi^{*}(j):u^{i}v^{-j}=:\psi(u)\psi^{*}(v):. \tag{7.2}$$

Using the definition (4.11) of normal ordering, this can also be written as

$$X(u,v)=\psi(u)\psi^{*}(v)-\frac{1}{1-v/u}, \tag{7.3}$$

where the power series $(1-v/u)^{-1}$ has only formal meaning;

$$\frac{1}{1-v/u}:=\sum_{k\geqslant 0}\left[\frac{v}{u}\right]^{k}. \tag{7.4}$$

In order to find an alternative expression for the fermionic fields $\psi(z)$ and $\psi^{*}(z)$, we consider the commutation relations of these fermionic fields with the bosonic oscillators.

Lemma 7.1.

$$[\hat{\alpha}(k),\psi(z)]=z^{k}\psi(z),$$
$$[\hat{\alpha}(k),\psi^{*}(z)]=-z^{k}\psi^{*}(z). \tag{7.5}$$

Proof. Using the definition (6.1) of the oscillators and the normal ordering prescription (4.10), we write

$$[\hat{\alpha}(k),\psi(j)]=-\sum_{l+k\leqslant 0}[\psi^{*}(l+k)\psi(l),\psi(j)]+\sum_{l+k>0}[\psi(l)\psi^{*}(l+k),\psi(j)]$$
$$=\sum_{l\in\mathbf{Z}}\delta_{l+k,j}\psi(l)=\psi(j-k).$$

With this formula the first relation of the lemma is clear. The second can be proved by Hermitian conjugacy of the first. □

The relations (7.5) can be seen as formal eigenvalue equations for the adjoint action of the oscillator algebra. It is easy to find solutions for (7.5); define

$$E^{(+)}(z) := \exp\left[-\sum_{k>0}\frac{1}{k}z^{-k}\alpha(k)\right],$$

$$E^{(-)}(z) := \exp\left[-\sum_{k<0}\frac{1}{k}z^{-k}\alpha(k)\right], \tag{7.6}$$

then one easily checks that the product $E^{(-)}(z)E^{(+)}(z)$ satisfies the first relation of (7.5) for all $k\neq 0$ (recall that $\hat{\alpha}(k)=\alpha(k)\ \forall k\neq 0$). In other words: the operator $Q(z) := E^{(-)}(z)^{-1}\psi(z)E^{(+)}(z)^{-1}$ commutes with all oscillators except with the zero mode;

$$[\hat{\alpha}(k), Q(z)] = \delta_{k0}Q(z). \tag{7.7}$$

With this remark it is clear that we can write

$$\psi(z) = Q(z)E^{(-)}(z)E^{(+)}(z). \tag{7.8}$$

For the Hermitian conjugate field one easily derives a similar expression;

$$\psi^*(z) = Q^*(z)E^{(-)}(z)^{-1}E^{(+)}(z)^{-1}. \tag{7.9}$$

It remains to determine $Q(z)$ and $Q^*(z)$. To do this, we use the commutation relations (5.12) of the fermionic fields with the Virasoro operators $\hat{H}_k = \hat{L}_k$.

Lemma 7.2. *We have:*

$$Q(z) = z^{\alpha(0)}Q,$$

$$Q^*(z) = z^{-\alpha(0)-1}Q^*, \tag{7.10}$$

where the Hermitian conjugate operators $Q, Q^*: \wedge^\infty\mathbb{C}^\infty \to \wedge^\infty\mathbb{C}^\infty$ *are determined by the relations*

$$Q\psi(z) = z^{-1}\psi(z)Q,$$

$$Q\psi^*(z) = z\psi^*(z)Q,$$

$$Q|0\rangle = \psi(1)|0\rangle,$$

$$Q^*|0\rangle = \psi^*(0)|0\rangle. \tag{7.11}$$

Moreover, these operators are unitary:

$$Q^* = Q^{-1}. \tag{7.12}$$

PROOF. Consider the commutator $[\hat{H}_0,\psi(z)]$. Using (5.12), the definition (6.12) of $\hat{L}_0 = \hat{H}_0$ and the expression (7.8) for $\psi(z)$, we find:

$$[\frac{1}{2}\sum_{k\in\mathbf{Z}} :\hat{\alpha}(-k)\hat{\alpha}(k):,\ Q(z)E^{(-)}(z)E^{(+)}(z)] = z\frac{d}{dz}(Q(z)E^{(-)}(z)E^{(+)}(z)). \tag{7.13}$$

With the commutation relations (6.3) for the oscillators it is easily seen that:

$$[\frac{1}{2}\sum_{k\in\mathbf{Z}} :\hat{\alpha}(-k)\hat{\alpha}(k):,\ E^{(-)}(z)E^{(+)}(z)] = z\frac{d}{dz}(E^{(-)}(z)E^{(+)}(z)). \tag{7.14}$$

Substituting this relation in (7.13), and recalling that $Q(z)$ commutes with all oscillators except with the zero mode, we find

$$[\frac{1}{2}\hat{\alpha}(0)^2,\ Q(z)] = z\frac{d}{dz}Q(z). \tag{7.15}$$

Using (7.7), this can be rewritten as

$$z\frac{d}{dz}Q(z) = \left[\hat{\alpha}(0) - \frac{1}{2}\right]Q(z) = \alpha(0)Q(z). \tag{7.16}$$

This differential equation is solved by

$$Q(z) = z^{\alpha(0)}Q, \tag{7.17}$$

where Q is some operator independent of z. The expression for $Q^*(z)$ can be derived from this formula by Hermitian conjugacy.

The proof of the other relations of the lemma is somewhat more technical. We refer to the paper [7] for a detailed derivation. □

Combining Lemma 7.2 with the relations (7.8-9), we find the following important corrollary:

COROLLARY 7.3.

$$\begin{aligned} \psi(z) &= z^{\alpha(0)}QE^{(-)}(z)E^{(+)}(z), \\ \psi^*(z) &= z^{-\alpha(0)-1}Q^{-1}E^{(-)}(z)^{-1}E^{(+)}(z)^{-1}. \end{aligned} \tag{7.18}$$

It is clear that Q is an operator that 'translates' the charge k sector into the charge $k+1$ sector. Therefore, we will call it a fermionic translation operator. In the expression for the vertex operator $X(u,v)$ the Q's cannot occur since the operators $\pi(E_{ij})$ map the charge k sector into itself. In fact, we find:

$$X(u,v) = (v/u)^{-\alpha(0)}E^{(-)}(u)E^{(+)}(u)E^{(-)}(v)^{-1}E^{(+)}(v)^{-1} - \frac{1}{1-v/u}I. \tag{7.19}$$

With a calculus of formal variables developed in [9] one derives:

$$E^{(+)}(u)E^{(-)}(v)^{-1}E^{(+)}(u)^{-1} = \frac{1}{1-v/u}E^{(-)}(v)^{-1}. \tag{7.20}$$

Using this result, we obtain the following theorem:

THEOREM 7.4. *The formal operator* $X(u,v):=\Sigma_{i,j\in\mathbb{Z}}\pi(E_{ij})u^iv^{-j}$ *can be expressed in the oscillators as follows:*

$$X(u,v)=\frac{(v/u)^{-\alpha(0)}}{1-v/u}E^{(-)}(u)E^{(-)}(v)^{-1}E^{(+)}(u)E^{(+)}(v)^{-1}-\frac{1}{1-v/u}I \tag{7.21}$$

$$=\frac{(v/u)^{-\alpha(0)}}{1-v/u}\exp\left[\sum_{k>0}\frac{1}{k}(u^k-v^k)\alpha(-k)\right]\exp\left[-\sum_{k>0}\frac{1}{k}(u^{-k}-v^{-k})\alpha(k)\right]-\frac{1}{1-v/u}I.$$

We stress that this is a remarkable result; it means that the action of the algebra $\hat{gl}(\infty)$ on the semi-infinite wedge space can be completely expressed in terms of the action of the subalgebra consisting of all oscillators. Combining this with the fact that the charge k sector is an irreducible $\hat{gl}(\infty)$ module, we conclude that $\wedge_k^\infty\mathbf{C}^\infty$ must remain irreducible under the action of this oscillator algebra. In other words: it is a bosonic Fock space. This result is usually derived using (a specialization of) the Weyl-Kac character formula (see, e.g. [10]). Here we have followed a different approach; we have derived the formula for $X(u,v)$ from the expressions (7.18) of the fermionic fields. The essential ingredient in the derivation of these expressions is Theorem 6.6: the result $Q(z)=z^{\alpha(0)}Q$ was found with the commutator $[\hat{H}_0,\psi(z)]$ and the fact that $H_0=\hat{L}_0$.

As was already mentioned in Section 6, we can construct an irreducible representation of the oscillator algebra on the ring of polynomials in all variables $x_1,x_2,\ldots$. Restriction to the charge zero sector, where $\alpha(0)=0$, yields formula (1.5).

8. MULTI-COMPONENT FERMIONS

Here we briefly discuss recent work on multicomponent fermionic constructions of the semi-infinite wedge space. The interested reader is referred to the paper [7] for more details and proofs.

Recall that the fermionic Fock space $\wedge^\infty\mathbf{C}^\infty$ can be constructed from the action of the creation and annihilation operators $\psi(k)$ and $\psi^*(k)$ on the vacuum $|0\rangle$. In Section 5 we have seen that the label k is related to the eigenvalues of the energy operator H_0. This means that we are dealing with fermions of only one type, e.g. an electron and its antiparticle, the positron. To describe a system of n different types of fermions, one introduces the Clifford algebra Cl on generators $\psi_i(k)$, $\psi_i^*(k)$, $1\leq i\leq n$, $k\in\mathbb{Z}$ with relations

$$\{\psi_i(k),\psi_j^*(l)\}=\delta_{ij}\delta_{kl}\quad\{\psi_i(k),\psi_j(l)\}=0=\{\psi_i^*(k),\psi_j^*(l)\}. \tag{8.1}$$

Let V be the unique irreducible Cl-module, which contains a vacuum vector $|0\rangle$ such that

$$\begin{aligned}&\psi_i(k)|0\rangle=0\quad\forall k\leq 0,\\&\psi_i^*(k)|0\rangle=0\quad\forall k>0.\end{aligned} \tag{8.2}$$

In fact this Clifford algebra is isomorphic to the one we had before. To see this, we define the relabelings:

$$\psi_i(k) := \psi(i+n(k-1)) \quad \psi_i^*(k) = \psi^*(i+n(k-1)). \tag{8.3}$$

From this and the defining relations (8.2) of the space V, it is clear that $V \cong \wedge^\infty \mathbf{C}^\infty$. So from a mathematical point of view the 1-component construction is equivalent to the n-component construction. It is also clear that the normal ordering prescription (4.10) can be transferred to the multi-component fermions as follows:

$$:\psi_i(k)\psi_j^*(l): \overset{def}{=} \begin{cases} \psi_i(k)\psi_j^*(l) & \text{if } l>0, \\ -\psi_j^*(l)\psi_i(k) & \text{if } l \leq 0. \end{cases} \tag{8.4}$$

The next step is to introduce fermionic fields $\psi_i(z)$ and $\psi_i^*(z)$ by:

$$\psi_i(z) := \sum_{k \in \mathbf{Z}} \psi_i(k) z^k,$$

$$\psi_i^*(z) := \sum_{k \in \mathbf{Z}} \psi_i^*(k) z^{-k}. \tag{8.5}$$

Bosons can be introduced in this picture analogous to (6.16);

$$\alpha_i(z) = \sum_{k \in \mathbf{Z}} \alpha_i(k) z^{-k} \overset{def}{=} :\psi_i(z)\psi_i^*(z):. \tag{8.6}$$

With this definition one easily verifies the oscillator commutation relations

$$[\alpha_i(k), \alpha_j(l)] = \delta_{ij}\delta_{k+l,0} I. \tag{8.7}$$

We are now ready to formulate the following analog of Lemma 7.2 and the expressions (7.18).

THEOREM 8.1.

$$\psi_i(z) = z^{\alpha_i(0)} Q_i \exp\left[-\sum_{k<0} \frac{1}{k} z^{-k} \alpha_i(k)\right] \exp\left[-\sum_{k>0} \frac{1}{k} z^{-k} \alpha_i(k)\right],$$

$$\psi_i^*(z) = z^{-\alpha_i(0)-1} Q_i^{-1} \exp\left[\sum_{k<0} \frac{1}{k} z^{-k} \alpha_i(k)\right] \exp\left[\sum_{k>0} \frac{1}{k} z^{-k} \alpha_i(k)\right]. \tag{8.8}$$

where the operators $Q_i : V \to V$ are defined by

$$\begin{aligned} Q_i|0> &= \psi_i(1)|0>, \\ Q_i\psi_i(k) &= \psi_i(k+1)Q_i, \\ Q_i\psi_i^*(k) &= \psi_i^*(k+1)Q_i, \\ Q_i\psi_j(k) &= -\psi_j(k)Q_i \quad \text{if } i \neq j, \\ Q_i\psi_j^*(k) &= -\psi_j^*(k)Q_i \quad \text{if } i \neq j. \end{aligned} \tag{8.9}$$

These operators satisfy:

$$\{Q_i, Q_j\} = 0 \quad \text{if } i \neq j. \tag{8.10}$$

The normal ordered products $:\psi_i(u)\psi_j^*(v):$ are again generating operators for the representation π of $\hat{gl}(\infty)$;

$$
\begin{aligned}
:\psi_i(u)\psi_j^*(v): &= \sum_{k,l\in\mathbf{Z}} :\psi_i(k)\psi_j^*(l): u^k v^{-l} \\
&= \sum_{k,l\in\mathbf{Z}} :\psi(i+n(k-1))\psi^*(j+n(l-1)): u^k v^{-l} \\
&= \sum_{k,l\in\mathbf{Z}} \pi(E_{i+n(k-1),j+n(l-1)})u^k v^{-l}. \qquad (8.11)
\end{aligned}
$$

For $i=j$ the product $:\psi_i(u)\psi_i^*(v):$ is simply the bosonic field $\alpha_i(z)$, while for $i \neq j$ we can derive after some formal calculations:

$$
:\psi_i(u)\psi_j^*(v): = Q_i Q_j^{-1} u^{1+\alpha_i(0)} v^{-\alpha_j(0)} \times
$$

$$
\exp\left[\sum_{k>0}\frac{1}{k}(u^k\alpha_i(-k)-v^k\alpha_j(-k))\right]\exp\left[-\sum_{k>0}\frac{1}{k}(u^{-k}\alpha_i(k)-v^{-k}\alpha_j(k))\right]. \qquad (8.12)
$$

It is important to notice that the Q's do not cancel in this formula, simply because there are n different types of them. This means that the charge k sector $\wedge_k^\infty \mathbf{C}^\infty$ is not irreducible anymore under the action of the algebra consisting of all oscillators $\alpha_i(k)$, $1 \leq i \leq n$, $k \in \mathbf{Z}$. Consequently, this oscillator algebra cannot be equivalent to the one consisting of the oscillators $\alpha(k)$, $k \in \mathbf{Z}$!

In fact there are many other inequivalent oscillator subalgebras in A_∞. They are associated to more general relabelings of the fermions than (8.3). To each equivalence class corresponds a vertex operator construction of the space $\wedge_k^\infty \mathbf{C}^\infty$. It is an open problem how to parametrize the equivalence classes of oscillator subalgebras of A_∞. In [7] we have systematically studied the oscillator subalgebras, which come from the representation theory of the affine algebra $\hat{gl}_n(\mathbf{C})$. In this context the algebra of $\alpha(k)$'s is the well-known 'principal' oscillator subalgebra of $\hat{gl}_n(\mathbf{C})$, while the algebra of $\alpha_i(k)$'s is the 'homogeneous' oscillator subalgebra of $\hat{gl}_n(\mathbf{C})$.

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